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Mixed Duality for Nonsmooth Multiobjective Fractional Programming without a Constraint Qualification

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1 Introduction and Preliminaries

For a nonempty subset C of $\mathbb{R}^n$, we denote C^*, the dual cone of C and defined by

$$C^* = \{u \in \mathbb{R}^n | u^T x \geqq 0, \ \ \forall \ \ x \in C\}.$$

Further, for $x^* \in C$, $N_C(x^*)$ denotes the normal cone to C at x^* defined by

$$N_C(x^*) = \{d \in \mathbb{R}^n | < d, x - x^* > \leqq 0, \ \forall \ x \in C\},$$

clearly, $(C - x^*)^* = -N_C(x^*)$.

We consider the following multiobjective nonsmooth fractional programming problem,

$$\begin{aligned} \text{(FP)} \quad \text{Minimize} \quad & \frac{f(x) + s(x|D)}{g(x)} \\ \text{subject to} \quad & h(x) \leqq 0, \\ & x \in C, \end{aligned}$$

where C is a convex set and $D_i, i = 1, \cdots, p$ are compact convex sets of $\mathbb{R}^n$ and f_i, g_i, h_j $i = 1, \cdots, p$, $j = 1, \cdots, m$ are real valued locally Lipschitz functions defined on C. The index sets are $P = \{1, 2, \cdots, p\}$, $M =$

$\{1, 2, \cdots, m\}$. We denote the feasible set $\{x \in C | h_j(x) \leqq 0,\ j = 1, \cdots, m\}$ by F. Let $I(x^*) = \{j \in M | h_j(x^*) = 0\}$ denote the index set of active constraints at x^*. We assume that for each $i \in P$, $f_i(x) + s(x|D_i) \geqq 0,\ g_i(x) > 0$.

The minimal index set of active constraints for F is denoted by

$$I^= = \{j \in M | x \in F \rightarrow h_j(x) = 0\}.$$

We also denote

$$I^<(x^*) = I(x^*) \setminus I^= = \{j \in I(x^*) | x \in F \text{ such that } h_j(x) < 0\}.$$

Definition 1.1 *A feasible solution x^* for (FP) is efficient for (FP) if and only if there is no other feasible x for (FP) such that*

$$\frac{f_{i_0}(x) + s(x|D_{i_0})}{g_{i_0}(x)} < \frac{f_{i_0}(x^*) + s(x^*|D_{i_0})}{g_{i_0}(x^*)}, \text{ for some } i_0 \in P,$$

and

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leqq \frac{f_i(x^*) + s(x^*|D_i)}{g_i(x^*)}, \ \forall\, i \in P.$$

Definition 1.2 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function. Then*
(i) *it is said to be generalized convex at x if for any y*

$$f(y) - f(x) \geqq <\xi, y - x>, \ \forall\, \xi \in \partial_c f(x),$$

(ii) *it is said to be generalized quasiconvex at x if for any y such that $f(y) \leqq f(x)$,*

$$<\xi, y - x> \leqq 0, \ \forall\, \xi \in \partial_c f(x),$$

(iii) *it is said to be generalized strictly quasiconvex at x if for any y such that $f(y) \leq f(x)$,*

$$<\xi, y - x> < 0, \ \forall\, \xi \in \partial_c f(x),$$

2 Optimality Conditions

We will establish necessary and sufficient optimality conditions for the fractional problem (FP).

We consider the following nonlinear programming problem:

$$
\begin{aligned}
(\mathrm{F}P_u) \quad \text{Minimize} \quad & (f_1(x) + s(x|D_1) - u_1 g_1(x), \\
& \cdots, f_p(x) + s(x|D_p) - u_p g_p(x)) \\
\text{subject to} \quad & h_j(x) \leqq 0, \; j \in M, \; x \in C,
\end{aligned}
$$

where for each $u = (u_1, \cdots, u_p) \in \mathbb{R}^n_+$.

Lemma 2.1 *If x^* is an efficient solution for* (FP), *then x^* is an efficient solution for* $(\mathrm{FP}_{\mathrm{u}^*})$ *where* $u^* = \frac{f_i(x^*)+s(x^*|D_i)}{g_i(x^*)}$.

We denote $\phi_i(x^*) = \frac{f_i(x^*)+s(x^*|D_i)}{g_i(x^*)}$.

Theorem 2.1 *If x^* be an efficient solution of* (FP) *and $h_j(\cdot), j \in M$ are generalized strictly quasiconvex at x^*, then there exist $\tau^* \in \mathbb{R}^p, \mu^* \in \mathbb{R}^m$, and $z_i^* \in \mathbb{R}^n, i \in P$ such that*

$$
\begin{aligned}
0 \in \textstyle\sum_{i=1}^p \tau_i^* [\partial_c (f_i(x^*) + (x^*)^T z_i) - \phi_i(x^*) \partial_c g_i(x^*)] \\
+ \textstyle\sum_{j \in I^<(x^*)} \mu_j^* \partial_c h_j(x^*) + N_c(x^*), \qquad (*) \\
\tau_i^* > 0, \; i \in P, \; \mu_j^* \geqq 0, \; j \in M, \; \textstyle\sum_{i=1}^p \tau_i^* = 1.
\end{aligned}
$$

Corollary 2.1 *Let x^* be an efficient solution for* (FP), *and $h_j, j \in M$ are generalized strictly quasiconvex at x^*, then there exist $\tau^* \in \mathbb{R}^p, \mu^* \in \mathbb{R}^m$ and $z_i^* \in D_i, i \in P$ such that*

$$0 \in \sum_{i=1}^{p} \tau_i^* [\partial_c (f_i(x^*) + (x^*)^T z_i) - \phi_i(x^*) \partial_c g_i(x^*)] + \sum_{j=1}^{m} \mu_j^* \partial_c h_j(x^*) + N_c(x^*),$$

$$(z_i^*)^T x^* = s(x^*|D_i), \ z_i^* \in D_i, \ i \in P,$$

$$\mu_j^* h_j(x^*) = 0, \ j \in M,$$

$$\tau_i^* > 0, \ i \in P, \ \mu_j^* \geqq 0, \ j \in M, \ \sum_{i=1}^{p} \tau_i^* = 1.$$

Theorem 2.2 *Let x^* be a feasible solution of* (FP) *and assume that the conditions* $(*)$ *hold at x^*. If all of the functions $f_i(\cdot), i = 1, \cdots, p$, $-g_i(\cdot)$ and $h_j(\cdot), j = 1, \cdots, m$ are generalized convex at x^*, then x^* is an efficient solution of* (FP).

3 Mixed Duality

We introduce a mixed type dual fractional programming problem and establish weak, strong and converse duality theorems.

$$\text{(FD) Maximize } \ \left(\frac{f_1(y) + y^T z_1 + \sum_{j \in M_1} \mu_j h_j(y)}{g_1(y)}, \cdots, \frac{f_p(y) + y^T z_p + \sum_{j \in M_1} \mu_j h_j(y)}{g_p(y)}\right)$$

$$\text{subject to } \ 0 \in \sum_{i \in P} g_i(y)(\partial_c(\tau_i(f_i(y) + y^T z_i)) + \tau_i \sum_{j \in M_1} \partial_c \mu_j h_j(y)) - \sum_{i \in P}(f_i(y)$$

$$+ y^T z_i + \sum_{j \in M_1} \mu_j h_j(y)) \partial_c \tau_i g_i(y) + \sum_{j \in M_2} \partial_c \mu_j h_j(y) + N_c(y),$$

$$\mu_j h_j(y) \geqq 0, j \in M_2, \ z_i \in D_i, i \in P,$$

$$\mu_j \geqq 0, j \in M, \ \tau_i > 0, \sum_{i \in M} \lambda_i = 1, y \in C.$$

where M_1 is a subset of $M = \{1, \cdots, m\}$, $M_2 = M \setminus M_1$. Assume that for each $i \in P$, $f_i(y) + y^T z_i + \sum_{j \in M_1} \mu_j h_j(y) \geqq 0, g_i(y) > 0$.

Theorem 3.1 (Weak Duality) *Let x be a feasible solution for problem* (FP) *and (y, z, τ, μ) be a feasible solution for problem* (FD). *If all of the functions $f_i(\cdot)$, $g_i(\cdot)$, $i \in P$, and $h_{M_1}(\cdot)$ are generalized convex at y, and $\mu_j h_j(\cdot)$, $j \in M_2$ are generalized quasiconvex at y, then the following cannot hold:*

$$\frac{f_{i_0}(x) + s(x|D_{i_0})}{g_{i_0}(x)} < \frac{f_{i_0}(y) + y^T z_{i_0} + \sum_{j \in M_1} \mu_j h_j(y)}{g_{i_0}(y)}, \text{ for some } i_0 \in P$$

and

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leqq \frac{f_i(y) + y^T z_i + \sum_{j \in M_1} \mu_j h_j(y)}{g_i(y)}, \ \forall\, i \in P.$$

Theorem 3.2 (Strong Duality) *Let x^* be an efficient solution of problem* (FP). *If the assumptions of Theorem 3.1 hold and $h_j(\cdot), j = 1, \cdots, m$ are generalized strictly quasiconvex, then there exist $\tau^* \in \mathbb{R}^p_+$, $z^* \in D_i$, $i \in P$ and $\mu^* \in \mathbb{R}^m_+$ such that $(x^*, z^*, \tau^*, \mu^*)$ is an efficient solution of* (FD), *$(x^*)^T z_i^* = s(x^*|D_i), i \in P$ and the optimal values of* (FP) *and* (FD) *are equal.*

Theorem 3.3 (Strict Converse Duality) *Let x^* and $(y^*, z^*, \tau^*, \mu^*)$ be efficient solutions of* (FP) *and* (FD), *respectively. If in addition the hypotheses of Theorem 3.2 hold, then $y^* = x^*$, that is, y^* solves* (FP) *and the objective values of* (FP) *and* (FD) *are equal.*

4 Special Cases

We give some special cases of our dual programming.

(1) If $D_i = \{0\},\ i = 1, \cdots, p$ and $I_1 = \emptyset$, then our mixed dual problem (FD) reduce to the Mond-Weir type problem (D_1) in Weir and Mond [13].

(2) If $D_i = \{0\},\ i = 1, \cdots, p$ and $I_2 = \emptyset$, then our mixed dual problem (FD) reduce to the Wolfe type problem (D_2) in Weir and Mond [13].

(3) If $D_i = \{0\},\ i = 1, \cdots, p$, then our mixed dual problem (FD) reduce to the mixed dual problem (MD) in Nobakhtian [10].

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